

# RESTRICTIONS ON THE PRIME TO $p$ FUNDAMENTAL GROUP OF A SMOOTH PROJECTIVE VARIETY

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**ABSTRACT.** The goal of this paper is to obtain restrictions on the prime to  $p$  quotient of the étale fundamental group of a smooth projective variety in characteristic  $p \geq 0$ . The results are analogues some theorems in the study of Kähler groups. Our first main result is that such groups are indecomposable under coproduct. The second gives a classification of the pro- $\ell$  parts of one relator groups in this class.

Our goal is obtain some analogues, over fields of characteristic  $p \geq 0$ , of a few of the known restrictions on the class of Kähler groups; we recall that a group is Kähler if it can be realized as the fundamental group of a compact Kähler manifold. In this paper, we replace the usual fundamental group by the maximal prime to  $p$  quotient of the étale fundamental group. We focus on this group, because it behaves most like its topological namesake. Let  $\mathcal{P}(p)$  denote the class of profinite groups that arise as prime to  $p$  fundamental groups of smooth projective varieties defined over an algebraically closed field of characteristic  $p$ . Our first main result implies, among other things, the indecomposability of groups in  $\mathcal{P}(p)$  under coproduct. This is an analogue of Gromov's theorem [G] in the Kähler setting. For the second main result, we show that if  $G \in \mathcal{P}(p)$  is the completion of a one relator group, then for almost every  $\ell$ , the pro- $\ell$  quotient  $G_\ell$  of  $G$  is isomorphic to the pro- $\ell$  fundamental group of a smooth projective curve. This is inspired by the recent classification of one relator Kähler groups of Biswas, Mahan [BM] and Kotschick [K], although the argument here is completely different. We deduce from the hard Lefschetz theorem that  $G_\ell$  is a Demushkin group for almost all  $\ell$ , then the result follows from the classification of such groups.

I would like to thank Jakob Stix for bringing Demushkin's work to my attention.

## 1. PRELIMINARIES

From the beginning, we fix an integer  $p$  which is either a prime number or 0. By a  $p'$ -group we will mean a finite group of order prime to  $p$  (or arbitrary when  $p = 0$ ), and by a pro- $p'$  group, we mean an inverse limit of  $p'$ -groups. The symbol  $\ell$  will always stand for a fixed or variable prime different from  $p$ . Given a pro-finite group  $G$ , let  $G_{p'}$ , respectively  $G_\ell$ , denote the maximal pro- $p'$ , respectively pro- $\ell$ , quotient of  $G$ . Given a discrete group  $G$ , we let

$$\hat{G} = \varprojlim_{G/N \text{ a } p'\text{-group}} G/N$$

denote the pro- $p'$  completion. So that  $\hat{\mathbb{Z}} = \prod_{\ell \neq p} \mathbb{Z}_\ell$ . Then  $\hat{G}_\ell$  can be identified with the pro- $\ell$  completion of  $G$ . Given a connected scheme  $X$ , let  $\pi_1^{et}(X)$  denote

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Grothendieck's étale fundamental group [SGA1], where we ignore the base point. This is the profinite group for which the category of finite sets with continuous action is equivalent to the category of étale covers of  $X$ . Let us write  $\pi_1^{p'}(X)$  and  $\pi_1^\ell(X)$  instead of  $\pi_1^{et}(X)_{p'}$  and  $\pi_1^{et}(X)_\ell$ . Given an algebraically closed field  $k$  of characteristic  $p$ , let  $\mathcal{P}(k)$  denote the class of pro- $p'$  groups which are isomorphic to  $\pi_1^{p'}(X)$ , where  $X$  is a smooth projective  $k$ -variety.  $\mathcal{P}(\mathbb{C})$  is the class of profinite completions of topological fundamental groups of complex smooth projective varieties. Set  $\mathcal{P}(p) = \mathcal{P}(\overline{\mathbb{F}}_p)$ , where  $\overline{\mathbb{F}}_p$  is the algebraic closure of the prime field of characteristic  $p$  (so that  $\overline{\mathbb{F}}_0 = \overline{\mathbb{Q}}$ ). There is no loss in focusing on this case because of the following fact.

**Proposition 1.1.** *If  $k$  is an algebraically closed field of characteristic  $p$ ,  $\mathcal{P}(k) = \mathcal{P}(p)$ .*

*Proof.* Clearly  $\mathcal{P}(p) \subseteq \mathcal{P}(k)$  because extension of scalars from  $\overline{\mathbb{F}}_p$  to  $k$  will not change the fundamental group [SGA1, exp X, cor 1.8]. Suppose that  $X$  is a smooth projective  $k$ -variety. It is defined over a finitely generated extension  $K$  of  $\overline{\mathbb{F}}_p$ , i.e. there exists a  $K$ -scheme  $X_K$  such that  $X = X_K \times_{\text{Spec } K} \text{Spec } k$ . Let  $S$  be variety defined over  $\overline{\mathbb{F}}_p$  with function field  $K$ . After shrinking  $S$ , if necessary, we can assume that there is a smooth projective morphism  $\mathcal{X} \rightarrow S$  with geometric generic fibre  $X_K$ . Choose an  $\overline{\mathbb{F}}_p$  rational point  $y_0 \in S$  and let  $\eta$  denote the geometric generic point. Then  $\pi_1^{p'}(\mathcal{X}_{y_0}) \cong \pi_1^{p'}(\mathcal{X}_\eta) \cong \pi_1^{p'}(X)$  by [SGA1, exp X, cor 3.9].  $\square$

**Lemma 1.2.** *If  $G \in \mathcal{P}(p)$  and  $H \subset G$  is open, then  $H \in \mathcal{P}(p)$ .*

*Proof.* If  $G = \pi_1^{p'}(X)$  then  $H = \pi_1^{p'}(Y)$  for some étale cover  $Y \rightarrow X$ .  $\square$

Given a finitely generated  $\mathbb{Z}_\ell$  module  $V$  with a continuous action by a profinite group  $G$ , we define

$$H^i(G, V) := \varprojlim_n H^i(G, V/\ell^n V)$$

$$H^i(G, V \otimes \mathbb{Q}_\ell) = H^i(G, V) \otimes \mathbb{Q}_\ell$$

This naive definition will suffice for our purposes, although there is one place where we are better off with the more subtle definition of Jannsen [J]. We summarize what we need about this in the following lemma.

**Lemma 1.3.** *Suppose that  $1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$  is an exact sequence of profinite groups with  $G$  topologically finitely generated and  $V$  a finitely generated  $\mathbb{Z}_\ell$ -module with continuous  $H$  action. Then there is the usual 5-term exact sequence of Hochschild-Serre*

$$0 \rightarrow H^1(H, V) \rightarrow H^1(G, V) \rightarrow H^0(H, H^1(K, V)) \rightarrow H^2(H, V) \rightarrow H^2(G, V)$$

*Proof.* Following Jannsen, we define  $H_{cont}^i(G, V)$  as the  $i$ th derived functor of

$$V \mapsto \varprojlim_n H^0(V/\ell^n V)$$

The Hochschild-Serre spectral sequence, and the resulting 5 term sequence, for  $H_{cont}^*$  can be constructed in the usual way. By [J, (2.1)], we have an exact sequence

$$(1) \quad 0 \rightarrow \varprojlim^1 H^{i-1}(G, V/\ell^n V) \rightarrow H_{cont}^i(G, V) \rightarrow H^i(G, V) \rightarrow 0$$

When  $i \leq 2$ , we claim that  $H^{i-1}(G, V/\ell^n V)$  is finite. For  $i = 1$  this is clear because  $V$  is finitely generated. So we have to show this for  $i = 2$ . We can find

an open normal subgroup  $G_1$  which acts trivially on  $V/\ell^n V$ . Then by Hochschild-Serre, we have an exact sequence

$$H^1(G/G_1, V/\ell^n V) \rightarrow H^1(G, V/\ell^n V) \rightarrow H^1(G_1, V/\ell^n V)$$

The finiteness of the middle group is a consequence of the finiteness of the outer groups. The first group is finite, because both  $G/G_1$  and  $V/\ell^n V$  are. The last group  $H^1(G_1, V/\ell^n V)$  is isomorphic to  $\text{Hom}_{\text{cont}}(G_1, V/\ell^n V)$ . By assumption,  $G$  contains a finitely generated dense subgroup  $\Gamma$ . The group  $\Gamma \cap G_1$  is easily seen to be finitely generated and dense in  $G_1$ . Therefore  $\text{Hom}_{\text{cont}}(G_1, V/\ell^n V)$  is finite and the claim is proved.

The Mittag-Leffler condition holds for  $H^{i-1}(G, V/\ell^n V)$  and  $i \leq 2$  by the previous claim. Therefore the  $\varprojlim^1$  in (1) vanishes, and so  $H^i(G, V) \cong H_{\text{cont}}^i(G, V)$  for  $i \leq 2$ . By the same argument,  $H^i(H, V) \cong H_{\text{cont}}^i(H, V)$ . So the 5-term sequence for  $H_{\text{cont}}^*$  can be identified with the one given in the statement of the lemma.  $\square$

**Lemma 1.4.** *If  $G$  is a profinite group and  $V$  an abelian pro- $\ell$  group, then*

$$(G/[G, G])_\ell \cong G_\ell/[G_\ell, G_\ell]$$

$$\text{Hom}(G, V) \cong \text{Hom}(G_\ell, V)$$

where  $\text{Hom}$  is the group of continuous homomorphisms.

*Proof.* It is enough to prove the first isomorphism, because the second is a consequence of it. Since  $(G/[G, G])_\ell$  is an abelian pro- $\ell$  group, the homomorphism  $G \rightarrow (G/[G, G])_\ell$  factors through the abelianization of the maximal pro- $\ell$  quotient  $G_\ell/[G_\ell, G_\ell]$ . So we have a homomorphism  $G_\ell/[G_\ell, G_\ell] \rightarrow (G/[G, G])_\ell$ . On the other hand, the map  $G \rightarrow G_\ell/[G_\ell, G_\ell]$  must factor through the maximal pro- $\ell$  quotient of the abelianization  $(G/[G, G])_\ell$ . This gives the inverse.  $\square$

**Proposition 1.5.** *Suppose that  $X$  is a connected scheme of finite type over  $k$ . Let  $G$  be a quotient of  $\pi_1^{\text{et}}(X)$  by a closed normal subgroup, such that  $G$  dominates  $\pi_1^\ell(X)$ . Given a finitely generated  $\mathbb{Z}_\ell$ -module  $V$  with continuous  $G$ -action, there exists a homomorphism to  $\ell$ -adic cohomology*

$$(2) \quad H^i(G, V) \rightarrow H^i(X, V)$$

This is compatible with the cup products

$$H^i(G, V) \otimes H^j(G, V') \rightarrow H^{i+j}(G, V \otimes V')$$

and

$$H^i(X, V) \otimes H^j(X, V') \rightarrow H^{i+j}(X, V \otimes V')$$

The map (2) is an isomorphism when  $i \leq 1$ .

*Proof.* We start by proving the analogous statements over  $\Lambda_n = \mathbb{Z}/\ell^n \mathbb{Z}$ , and then take the limit. We indicate two different constructions of the map; the first is simpler, but second gives more, and so it is the one that we use. First of all, both  $H^i(G, -)$  and  $H^i(X, -)$  are  $\delta$ -functors from the category of discrete  $\Lambda_n[G]$ -modules to abelian groups, with the first universal in the sense of [Gr]. By the connectedness assumption  $H^0(G, V) \cong H^0(X, V)$ . Thus we get a map

$$H^*(G, -) \rightarrow H^*(X, -)$$

of  $\delta$ -functors. Compatibility with cup products can be proved in principle by dimension shifting and induction, but it seems simpler to give an alternate interpretation.

Suppose that  $Y \rightarrow X$  is a Galois étale cover with Galois group  $H$  a quotient of  $G$  by an open normal subgroup. Then we have an isomorphism of simplicial schemes

$$\text{cosk}(Y \rightarrow X)_\bullet \cong (Y \times EH_\bullet)/G$$

where  $\text{cosk}(Y \rightarrow X)_\bullet$  is the simplicial scheme

$$\dots Y \times_X Y \rightrightarrows Y$$

and  $EH_\bullet \rightarrow BH_\bullet$  is a simplicial model for the universal  $H$ -bundle over the classifying space (cf [D1, §5.1, 6.1] or [M1, pp 99-100]). The projection  $\text{cosk}(Y \rightarrow X)_\bullet \rightarrow EH_\bullet/G = BH_\bullet$  induces a map from the bar complex  $C^\bullet(H, V)$  with coefficients in an  $H$ -module  $V$  to the Čech complex  $\check{C}^\bullet(Y \rightarrow X, V)$ . Thus we obtain maps

$$H^*(G, V) \rightarrow H^*(H, V) \rightarrow H^*(X, V)$$

The compatibility with cup products now follows easily from the standard simplicial formulas for them [M1, p 172].

We have already seen that the map (2) is an isomorphism when  $i = 0$ . We next prove that it is an isomorphism when  $i = 1$ . First, suppose that  $V$  is a finite  $\Lambda_n$ -module with trivial  $G$ -action. Then have an isomorphism

$$H^1(G, V) \cong \text{Hom}(G, V)$$

On the other hand, we have

$$H^1(X, V) \cong \text{Hom}(\pi_1^{\text{ét}}(X), V)$$

because both groups classify  $V$ -torsors [M1, pp 121-123]. By lemma 1.4, we can also identify

$$\text{Hom}(\pi_1^{\text{ét}}(X), V) \cong \text{Hom}(G, V)$$

Now suppose that  $V$  is a nontrivial finite  $\Lambda_n[G]$ -module. Let  $\pi : Y \rightarrow X$  be an étale cover such that  $\pi^*V$  is trivial. We can assume that  $\pi$  is a Galois cover, with Galois group  $H$  a quotient of  $G$ . Set  $K = \pi_1^{\text{ét}}(Y)$ . Then  $K$  acts trivially on  $\pi^*V$ . It follows that

$$(3) \quad H^i(K, V) \cong H^i(Y, \pi^*V), \quad i = 0, 1$$

and this isomorphism is compatible with the  $H$  action. Then Hochschild-Serre gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(H, H^0(K, V)) & \longrightarrow & H^1(G, V) & \longrightarrow & H^0(H, H^1(K, V)) \longrightarrow H^2(H, H^0(K, V)) \\ & & \downarrow \cong & & \downarrow f & & \downarrow \cong \\ 0 & \longrightarrow & H^1(H, H^0(Y, V)) & \longrightarrow & H^1(X, V) & \longrightarrow & H^0(H, H^1(Y, V)) \longrightarrow H^2(H, H^0(X, V)) \end{array}$$

with exact rows. The maps labeled by  $\cong$  are isomorphisms by (3). Thus  $f$  is an isomorphism by the 5-lemma.

To summarize, we have canonical multiplicative homomorphisms

$$H^i(G, V) \rightarrow H^i(X, V)$$

for  $\Lambda_n[G]$ -modules, which are isomorphisms for  $i \leq 1$ . The proposition follows by taking the inverse limit over  $n$ .  $\square$

We will apply the last proposition in the two cases  $G = \pi_1^{p'}(X)$  and  $G = \pi_1^\ell(X)$ . It is worth remarking that when  $V = \mathbb{Q}_\ell$ , we have  $H^1(\pi_1^{p'}(X), V) \cong H^1(\pi_1^\ell(X), V)$ , but there is no reason to expect this for higher cohomology.

We have the following basic finiteness property.

**Theorem 1.6** (Raynaud ). *Any element of  $P(p)$  is topologically finitely presented.*

*Proof.* This follows from [R, thm 2.3.1, rem 2.3.2].  $\square$

The analogous statement for Kähler groups is a well known consequence of the finite triangulability of compact manifolds. We wish to point out that topological finite presentability does not preclude some fairly wild examples such as  $\prod_{\ell \neq p} \mathbb{Z}^\ell$ . However, such examples cannot lie in  $\mathcal{P}(p)$ .

**Proposition 1.7.** *If  $G \in P(p)$ , then  $G/[G, G]$  is the product of a finite abelian group with  $\hat{\mathbb{Z}}^b = \prod_{\ell \neq p} \mathbb{Z}_\ell^b$  where  $b$  is an even integer.*

*Proof.* We can decompose  $G/[G, G] = \prod_{\ell \neq p} \mathbb{Z}_\ell^{b_\ell} \times A_\ell$ , where  $A_\ell$  is a finite abelian  $\ell$ -group. We have to show that  $b_\ell$  is constant and that  $A_\ell = 0$  for  $\ell \gg 0$ . The Kummer sequence [M1, p 66] gives an isomorphism

$$\mathrm{Hom}(\prod \mathbb{Z}_\ell^{b_\ell} \times A_\ell, \mathbb{Z}_\ell) \cong T_\ell \mathrm{Pic}(X) = T_\ell \mathrm{Pic}^0(X)_{red}$$

where we identify  $\mathbb{Z}_\ell \cong \mathbb{Z}_\ell(1)$ . For the last equality, we use the exact sequence

$$0 \rightarrow \mathrm{Pic}^0(X) \rightarrow \mathrm{Pic}(X) \rightarrow NS(X) \rightarrow 0$$

and the fact that the Neron-Severi group  $NS(X)$  is finitely generated. Since  $\mathrm{Pic}^0(X)_{red}$  is an abelian variety, it follows that  $b_\ell = b = 2 \dim \mathrm{Pic}^0(X)_{red}$  (see for example [M2, thm 15.1]).

Again by Kummer, we have an isomorphism

$$\mathrm{Hom}(\prod \mathbb{Z}_\ell^{b_\ell} \times A_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong \ell\text{-torsion subgroup of } \mathrm{Pic}(X)$$

Since  $NS(X)$  is finitely generated, the  $\ell$ -torsion subgroups of  $\mathrm{Pic}(X)$  and  $\mathrm{Pic}^0(X)$  coincide for all  $\ell \gg 0$ . The  $\ell$ -torsion subgroup of  $\mathrm{Pic}^0(X)$  is isomorphic to  $(\mathbb{Z}/\ell\mathbb{Z})^b$ . Therefore for  $\ell \gg 0$ , we must have  $\mathrm{Hom}(A_\ell, \mathbb{Z}/\ell\mathbb{Z}) = 0$  which implies that  $A_\ell = 0$ .  $\square$

## 2. CONSEQUENCES OF HARD LEFSCHETZ

By far the simplest restriction on Kähler groups is what we will refer to as the parity test: a finitely generated  $\Gamma$  cannot be Kähler unless  $\mathrm{rank}(\Gamma/[\Gamma, \Gamma])$  is even. This is a consequence of the Hodge decomposition. Proposition 1.7 gives an analogue in our setting. It is convenient to record the relevant part of it as a corollary.

**Corollary 2.1.** *If  $G \in \mathcal{P}(p)$ ,  $\mathrm{rank} G_\ell/[G_\ell, G_\ell]$  is a fixed even integer for each  $\ell \neq p$ .*

The following fact, which refines the previous result, was first observed by Johnson and Rees [JR] in the Kähler group setting.

**Theorem 2.2.** *Let  $G \in \mathcal{P}(p)$ , and let  $H$  be a quotient of  $G$  by a closed normal subgroup such that  $H$  dominates  $G_\ell$ . Suppose that  $\rho : H \rightarrow O_n(\mathbb{Q}_\ell)$  is an orthogonal representation such that  $\rho(H)$  is finite, and let  $V$  be the corresponding  $H$ -module with quadratic form  $q : V \otimes V \rightarrow \mathbb{Q}_\ell$ . Then there exists a linear map  $\lambda : H^2(H, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell$  such that  $\lambda(q(\alpha \cup \beta))$  defines a symplectic pairing on  $H^1(H, V)$ .*

*Proof.* Suppose that  $G = \pi_1^{p'}(X)$ , where  $X$  is an  $n$  dimensional smooth projective variety. Fix an ample line bundle  $\mathcal{O}_X(1)$ , and let  $L$  denote the corresponding Lefschetz operator. We claim that

$$(4) \quad H^1(X, V) \times H^1(X, V) \xrightarrow{q \circ \cup} H^2(X, \mathbb{Q}_\ell) \xrightarrow{L^{n-1}} H^{2n}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$$

gives a nondegenerate symplectic pairing. When  $\rho$  is trivial,  $(V, q)$  is a sum of  $n$  copies of  $\mathbb{Q}_\ell$  with the standard pairing, and the claim follows from the hard Lefschetz theorem for étale cohomology [D, 4.1.1]. In general, let  $\pi : Y \rightarrow X$  be a Galois étale cover with Galois group  $K$  such that  $\pi^*V$  is trivial. We can decompose  $\pi_*\pi^*V = V \oplus V'$  under the  $K$  action. Let  $p : \pi_*\pi^*V \rightarrow V$  denote the projection. We equip  $Y$  with the Lefschetz operator corresponding to  $\pi^*\mathcal{O}_X(1)$ . Then the pairing (4) is obtained by applying  $p$  to the nondegenerate pairing

$$H^1(Y, \pi^*V) \times H^1(Y, \pi^*V) \longrightarrow H^2(Y, \mathbb{Q}_\ell) \xrightarrow{L^{n-1}} H^{2n}(Y, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$$

and the claim follows for general  $V$  with finite monodromy.

By proposition 1.5, we have a commutative diagram

$$\begin{array}{ccc} H^1(H, V) \times H^1(H, V) & \longrightarrow & H^2(H, \mathbb{Q}_\ell) \\ \downarrow \cong & & \downarrow \iota \searrow \lambda \\ H^1(X, V) \times H^1(X, V) & \longrightarrow & H^2(X, \mathbb{Q}_\ell) \xrightarrow{L^{n-1}} \mathbb{Q}_\ell \end{array}$$

where  $\lambda$  is defined as  $L^{n-1} \circ \iota$ . □

*Second proof of corollary 2.1.*  $H^1(G, \mathbb{Q}_\ell)$  carries a symplectic pairing, so it must be even dimensional. □

The theorem itself gives more subtle information than the parity test. For example, we have the following consequence.

**Proposition 2.3.** *Suppose that*

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$$

*is an extension of pro- $p'$  groups such that  $H^1(H, \mathbb{Q}_\ell) \neq 0$  and the transgression  $\text{Hom}_H(K, \mathbb{Q}_\ell) \rightarrow H^2(H, \mathbb{Q}_\ell)$  is an isomorphism. Then  $G \notin \mathcal{P}(p)$*

*Proof.* From the Hochschild-Serre sequence (lemma 1.3)

$$0 \rightarrow H^1(H, \mathbb{Q}_\ell) \xrightarrow{\alpha} H^1(G, \mathbb{Q}_\ell) \rightarrow \text{Hom}_H(K, \mathbb{Q}_\ell) \xrightarrow{\sim} H^2(H, \mathbb{Q}_\ell) \xrightarrow{\beta} H^2(G, \mathbb{Q}_\ell)$$

we conclude that  $\alpha$  is an isomorphism and  $\beta = 0$ . Therefore

$$\cup : \wedge^2 H^1(G, \mathbb{Q}_\ell) \rightarrow H^2(G, \mathbb{Q}_\ell)$$

is zero, because it factors through  $\beta$ . Thus  $G \notin \mathcal{P}(p)$  by theorem 2.2. □

The conditions of the proposition are easy to check for the following example.

**Corollary 2.4.** *The completion of the Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \hat{\mathbb{Z}} \right\}$$

is not in  $\mathcal{P}(p)$

A more general class of examples where the proposition applies comes from generalized universal central extensions. Given a pro- $p'$  group  $H$ , we have a (generally noncanonical) central extension

$$(5) \quad 0 \rightarrow H_2(H, \hat{\mathbb{Z}}) \rightarrow G \rightarrow H \rightarrow 1$$

with extension class lifting the identity under the surjection

$$H^2(H, H_2(H, \hat{\mathbb{Z}})) \rightarrow \text{Hom}(H_2(H, \hat{\mathbb{Z}}), H_2(H, \hat{\mathbb{Z}}))$$

Here the (co)homologies are defined by taking inverse limits of the usual groups with coefficients in  $\mathbb{Z}/n\mathbb{Z}$ . Transgression gives an isomorphism, so that:

**Corollary 2.5.** *If  $H^1(H, \mathbb{Q}_\ell)$  is nonzero, then the group  $G$  of the above extension (5) is not in  $\mathcal{P}(p)$ .*

The last statement should be compared with [Re, p 717, cor].

### 3. FREE PRODUCTS

Given two pro- $p'$  groups  $G_1$  and  $G_2$ , their coproduct in the category of pro- $p'$  groups exists [RZ, §9.1]. We denote it by  $G_1 \hat{*} G_2$ . It is closely related to the usual free product  $*$ .

**Lemma 3.1.** *Given discrete groups  $G_i$ ,  $\widehat{G_1 * G_2} \cong \hat{G}_1 \hat{*} \hat{G}_2$ .*

*Proof.* [RZ, 9.1.1]. □

The completion  $\hat{F}^r$  of the usual free group on  $r$  generators is a free pro- $p'$  group. It can also be expressed as a coproduct

$$\hat{F}^r = \hat{\mathbb{Z}} \hat{*} \dots \hat{*} \hat{\mathbb{Z}} \quad (r \text{ factors})$$

In [ABR], it is shown that a Kähler group cannot be an extension of a group with infinitely many ends by a finitely generated group. We observe that any nontrivial free product other than  $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$  has infinitely many ends. Since we do not (yet) have a theory of ends in the profinite setting, we give a slightly weaker statement involving the aforementioned class. On the other hand, the hypothesis on the kernel can be relaxed slightly.

**Theorem 3.2.** *Let  $p \neq 2$ . Suppose that we have an extension of pro- $p'$  groups*

$$(6) \quad 1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$$

*such that*

- (a)  $(K/[K, K])_2 / (\text{torsion})$  is a finitely generated  $\mathbb{Z}_2$ -module, and
- (b)  $H$  is a nontrivial coproduct other than  $(\mathbb{Z}/2\mathbb{Z}) \hat{*} (\mathbb{Z}/2\mathbb{Z})$ .

*Then  $G \notin \mathcal{P}(p)$*

*Proof.* Suppose that  $G$  fits into the exact sequence (6) with  $H = H_1 \hat{*} H_2$ , where  $H_i$  are nontrivial and not both of order 2. We will show that  $G$  cannot lie in  $\mathcal{P}(p)$ . We first make a reduction to the case where  $H$  is of the form  $J \hat{*} \hat{F}^2$ . Choose nontrivial finite quotients  $Q_i$  of  $H_i$  such that  $|Q_i| > 2$  for some  $i$ . Let  $L \subset H$  be the kernel of the projection  $H \rightarrow Q_1 \times Q_2$ . Then by the profinite version of the Kurosh subgroup theorem [RZ, 9.1.9], we see that  $L \cong J \hat{*} \hat{F}^2$  for some group  $J$ . It suffices to prove that the preimage  $\tilde{G}$  of  $L$  in  $G$  is not in  $\mathcal{P}(p)$  by lemma 1.2. Since it fits into an extension

$$1 \rightarrow K \rightarrow \tilde{G} \rightarrow L \rightarrow 1$$

we may replace  $G$  by  $\tilde{G}$  and  $H$  by  $L$ .

From the exact sequence (6), we get a continuous action of  $H$  on  $K/[K, K]$ . Therefore  $M = (K/[K, K])_2 \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$  is a finite dimensional representation of  $H$ . With respect to the factor  $\hat{F}^2 = \hat{\mathbb{Z}} \hat{*} \hat{\mathbb{Z}}$  of  $H$ , we get two actions of  $\hat{\mathbb{Z}}$  on  $M$  that we refer to as the first and second. Let  $\{\xi_1, \dots, \xi_n\}$  be the (possibly empty) set of one dimensional characters of  $\hat{F}^2$  corresponding to one dimensional subquotients of  $M$ . We may suppose that  $\xi_1, \dots, \xi_m$  are the characters among these with finite order. Let  $S \subset \hat{F}^2$  be the intersection of kernels of  $\xi_1, \dots, \xi_m$ . The group  $S$  is necessarily of the form  $\hat{F}^r$  with  $r \geq 2$  [RZ, 3.6.2]. After replacing  $H$  by  $J \hat{*} S = (J \hat{*} \hat{F}^{r-2}) \hat{*} \hat{F}^2$ ,  $J$  by  $J \hat{*} \hat{F}^{r-2}$  and  $G$  by the preimage of the new  $H$  in the old  $G$ , we may assume that all the characters  $\xi_i$  are either trivial or of infinite order. Let  $\xi'_i$  denote the restrictions of  $\xi_i$  to the first factor of  $\hat{F}^2 = \hat{\mathbb{Z}} \hat{*} \hat{\mathbb{Z}}$ . Then the sign character  $\sigma : \hat{\mathbb{Z}} \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Q}_2^*$ ,

$$\sigma(x) = \begin{cases} +1 & \text{if } x \in 2\hat{\mathbb{Z}} \\ -1 & \text{otherwise} \end{cases}$$

is not in  $\{\xi'_1, \dots, \xi'_n\}$ . Let  $\chi_1 = \sigma$  and  $\chi_2 \in \{1, \sigma\}$ , where the precise choice will be determined below. Let  $V = \mathbb{Q}_2$  denote the  $H = J \hat{*} \hat{\mathbb{Z}} \hat{*} \hat{\mathbb{Z}}$  module where  $J$  acts trivially and the two  $\hat{\mathbb{Z}}$  factors act through  $\chi_1$  and  $\chi_2$  respectively. We note that  $V$  is orthogonal, so that we can apply theorem 2.2 when the time comes.

We now compute  $\dim H^1(G, V)$ . From the Hochschild-Serre sequence (lemma 1.3), we obtain the exact sequence

$$(7) \quad 0 \rightarrow H^1(H, V) \rightarrow H^1(G, V) \rightarrow H^0(H, H^1(K, V))$$

We can identify

$$H^0(H, H^1(K, V)) \cong H^0(H, \text{Hom}(K/[K, K], V)) \cong \text{Hom}_H(M, V)$$

Since we chose  $\chi_1 \notin \{\xi'_1, \dots, \xi'_n\}$ , the latter space is zero. Therefore, by (7) we obtain an isomorphism

$$H^1(G, V) \cong H^1(H, V)$$

By an appropriate Mayer-Vietoris sequence [RZ, prop 9.2.13], we see that

$$H^1(H, V) \cong H^1(J, V) \oplus H^1(\hat{\mathbb{Z}}, \mathbb{Q}_{2, \chi_1}) \oplus H^1(\hat{\mathbb{Z}}, \mathbb{Q}_{2, \chi_2})$$

where the subscripts  $\chi_i$  indicate the action. The middle group on the right vanishes because  $\chi_1$  was nontrivial. By choosing  $\chi_2$  to be trivial or not according to the parity of  $\text{rank}(J/[J, J])$ , we see that the right side can be made to have odd dimension. Therefore  $G$  cannot be the pro- $p'$  fundamental group of a smooth projective variety by theorem 2.2.  $\square$



**Corollary 3.3.** *A group in  $\mathcal{P}(p)$  cannot decompose as a coproduct of nontrivial pro- $p'$  groups, and in particular it cannot be free.*

*Proof.* The only case not covered by the last theorem is  $(\mathbb{Z}/2\mathbb{Z}) \hat{*} (\mathbb{Z}/2\mathbb{Z})$ , but since this contains  $\hat{\mathbb{Z}}$  as an open subgroup, it is ruled out by corollary 2.1.  $\square$

**Corollary 3.4.** *Suppose that  $G$  satisfies all of the assumptions of the theorem but with (a) replaced by*

(a')  *$K$  is topologically finitely generated.*

*Then  $G \notin \mathcal{P}(p)$ .*

*Proof.* (a') implies (a).  $\square$

**Corollary 3.5.** *Suppose that  $1 \rightarrow K \rightarrow G \rightarrow H_1 * H_2 \rightarrow 1$  is an exact sequence of discrete groups, with  $K$  finitely generated and  $\hat{H}_i$  nontrivial and not both of order 2. Then  $\hat{G} \notin \mathcal{P}(p)$ .*

*Proof.* By [RZ, prop 3.2.5] and lemma 3.1, we have an exact sequence

$$\hat{K} \rightarrow \hat{G} \xrightarrow{f} \hat{H}_1 \hat{*} \hat{H}_2 \rightarrow 1$$

Therefore  $\ker f$  is topologically finitely generated.  $\square$

As an illustration of the use of this theorem, we show that the pure braid group does not lie in this class. This is a direct translation of the argument in [A] for showing that braid groups are not Kähler. Recall that  $B_n$  is given by generators  $s_1, \dots, s_{n-1}$  with relations  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  and  $s_i s_j = s_j s_i$  if  $|i - j| > 1$ . This maps to the symmetric group  $S_n$  by sending  $s_i \mapsto (i \ i + 1)$ . The kernel is the pure braid group  $P_n$ . More geometrically,  $P_n$  is the fundamental group of the configuration space of  $n$  distinct ordered points in the plane.

**Proposition 3.6.**  $\hat{P}_n \notin \mathcal{P}(p)$ .

*Proof.* We have  $P_2 = \mathbb{Z}$ , so  $\hat{P}_2 \notin \mathcal{P}(p)$  by corollary 2.1. The group  $B_3$  is generated by  $a = s_1 s_2 s_1$  and  $b = s_1 s_2$  with the relation  $a^2 = b^3$ . There is a surjective homomorphism from  $f : B_3 \rightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$  which sends  $a$  and  $b$  to the generators of  $\mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}/3\mathbb{Z}$  respectively. The kernel of  $f$  is the cyclic group generated by  $a^2 \in P_3$ . Thus we have an extension

$$0 \rightarrow \mathbb{Z} \rightarrow P_3 \rightarrow f(P_3) \rightarrow 1$$

By Kurosh's subgroup theorem [S3, §5.5] the image  $f(P_3)$  is a free product of a nonabelian free group and some additional factors. Therefore  $\hat{P}_3 \notin \mathcal{P}(p)$  by corollary 3.5. When  $n > 3$ , projection of the configuration spaces gives a fibration resulting a surjective homomorphism  $P_n \rightarrow P_3$  with finitely generated kernel. It follows that  $P_n \rightarrow f(P_3)$  is again surjective with finitely generated kernel. So once again corollary 3.5 shows that  $\hat{P}_n \notin \mathcal{P}(p)$ .  $\square$

## 4. ONE RELATOR GROUPS

Recently, Biswas-Mahan [BM] and Kotschick [K] have classified one relator Kähler groups: they are all fundamental groups of one dimensional compact orbifolds with at most one orbifold point. In more explicit terms, such a group would be of the form

$$\Gamma_{g,m} = \begin{cases} \langle x_1, \dots, x_{2g} \mid ([x_1, x_{g+1}] \dots [x_g, x_{2g}])^m \rangle & \text{if } g > 0 \\ \mathbb{Z}/m\mathbb{Z} = \langle x \mid x^m \rangle & \text{if } g = 0 \end{cases}$$

(Note that both [BM, K] classify infinite one relator Kähler groups, but the statement as given above is an immediate consequence.) We prove a pro- $\ell$  version for large  $\ell$  assuming that the relation lies in the commutator subgroup. To reconcile the statement below with the one just given, observe that  $(\hat{\Gamma}_{g,m})_\ell \cong (\hat{\Gamma}_{g,1})_\ell$  when  $\ell$  is coprime to  $m$ .

**Theorem 4.1.** *Suppose that  $G \in \mathcal{P}(p)$  is the pro- $p'$  completion of a discrete one-relator group. Then there exists an explicit finite set  $S$  of primes such that if  $\ell \notin S$ , the maximal pro- $\ell$  quotient  $G_\ell$  of  $G$  is isomorphic to the pro- $\ell$  completion of the genus  $g$  surface group  $\Gamma_{g,1}$  where  $g = \frac{1}{2} \dim H^1(G, \mathbb{Q}_\ell)$ .*

Before giving the proof, we need the following version of Stallings' theorem [St].

**Lemma 4.2.** *If  $f : G \rightarrow H$  is a continuous homomorphism of pro- $\ell$  groups such that the induced map  $H^i(H, \mathbb{Z}/\ell\mathbb{Z}) \cong H^i(G, \mathbb{Z}/\ell\mathbb{Z})$  is an isomorphism for  $i = 1$  and an injection for  $i = 2$ , then  $f$  is an isomorphism.*

*Proof.* The surjectivity of  $f$  follows from [S2, I prop 23], so it remains to check injectivity. Define the  $\ell$ -central series by  $C^0(G) = G$ , and  $C^{n+1}(G) = [G, C^n(G)]C^n(G)^\ell$ . We claim that  $f$  induces an isomorphism  $G/C^n(G) \rightarrow H/C^n(H)$ . The injectivity of  $f$  will follow from this claim because one has  $\bigcap C^n(G) = \{1\}$ . The proof of the claim is essentially identical to the argument in [St] in dual form; nevertheless we give it for completeness. This proof goes by induction. The initial case  $n = 1$  follows from the isomorphism  $H^1(H, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(G, \mathbb{Z}/\ell\mathbb{Z})$ . For the induction step we use the commutative diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C^n H / C^{n+1} H & \longrightarrow & H / C^{n+1} H & \longrightarrow & H / C^n H & \longrightarrow & 1 \\ & & \downarrow \gamma & & \downarrow f_{n+1} & & \downarrow f_n & & \\ 1 & \longrightarrow & C^n G / C^{n+1} G & \longrightarrow & G / C^{n+1} G & \longrightarrow & G / C^n G & \longrightarrow & 1 \end{array}$$

We have to show that  $f_{n+1}$  is an isomorphism assuming this for  $f_n$ . It is enough to check that  $\gamma$  is an isomorphism. We have a diagram coming from Hochschild-Serre,

$$\begin{array}{ccccccccc} H^1(H/C^n) & \longrightarrow & H^1(H) & \longrightarrow & \text{Hom}(C^n H, \mathbb{Z}/\ell) & \longrightarrow & H^2(H/C^n) & \longrightarrow & H^2(H) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma^* & & \downarrow \delta & & \downarrow \epsilon \\ H^1(G/C^n) & \longrightarrow & H^1(G) & \longrightarrow & \text{Hom}(C^n G, \mathbb{Z}/\ell) & \longrightarrow & H^2(G/C^n) & \longrightarrow & H^2(G) \end{array}$$

The hypotheses, including the induction hypothesis, implies that  $\alpha, \beta, \delta$  are isomorphisms, and  $\epsilon$  is injective. Therefore  $\gamma^*$  is an isomorphism by the 5-lemma. This implies that  $\gamma$  is an isomorphism. □

*Proof of theorem 4.1.* Let  $G$  be the completion of the quotient of the free group on  $d$  letters  $F = F^d$  by the normal subgroup  $R$  generated by single element  $r \in F$  with  $r \neq 1$ .

We have two cases. The first case is where  $r \in [F, F]$ . The associated graded of  $F$  with respect to the lower central series

$$Gr(F) = F/[F, F] \oplus [F, F]/[F, [F, F]] \oplus \dots$$

is a graded Lie algebra over  $\mathbb{Z}$  with Lie bracket induced by the commutator  $[Lz]$ . Let  $x_1, \dots, x_d$  denote generators of  $F$ . The first summand  $F/[F, F]$  is a free  $\mathbb{Z}$ -module freely generated by the classes  $\bar{x}_i$  of  $x_i$ , and the next summand  $[F, F]/[F, [F, F]]$  is freely generated by  $[\bar{x}_i, \bar{x}_j]$ , with  $i < j$ . Thus we can expand the class  $\bar{r} \in [F, F]/[F, [F, F]]$  of  $r$  as  $\bar{r} = \sum a_{ij} [\bar{x}_i, \bar{x}_j]$  with  $a_{ij} \in \mathbb{Z}$ . We extend  $(a_{ij})$  to a skew symmetric matrix by setting  $a_{ji} = -a_{ij}$  and  $a_{ii} = 0$ . By [RZ, 3.2.5], we have an exact sequence

$$\hat{R}_\ell \rightarrow \hat{F}_\ell \rightarrow G_\ell \rightarrow 1$$

Thus  $\hat{G}_\ell$  is also a one relator group in the topological sense, and therefore  $\dim_{\mathbb{F}_\ell} H^2(G_\ell, \mathbb{Z}/\ell\mathbb{Z}) = 1$  by [S2, cor, p 31]. We can also conclude that

$$G_\ell/[G_\ell, G_\ell] \cong \hat{F}_\ell/[\hat{F}_\ell, \hat{F}_\ell] \cong \mathbb{Z}_\ell^d$$

Thus  $d = \dim_{\mathbb{F}_\ell} H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z})$  is the minimal number of generators of  $G_\ell$ . From theorem 2.2, it follows that  $d = 2g$  for some integer  $g$  and  $H^2(G_\ell, \mathbb{Q}_\ell) \neq 0$ . Therefore  $H^2(G_\ell, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$  and the cup product pairing

$$H^1(G_\ell, \mathbb{Z}_\ell) \times H^1(G_\ell, \mathbb{Z}_\ell) \rightarrow H^2(G_\ell, \mathbb{Z}_\ell)$$

is nondegenerate. By an argument identical to the proof of [L, prop 3], we see that this pairing is represented by the matrix  $(a_{ij})$ . Let  $S$  denote the union of  $\{2, p\}$  and the set of all prime factors of the  $a_{ij}$ . Then we can reduce modulo  $\ell \notin S$  to obtain a nondegenerate cup product pairing

$$H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z}) \times H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H^2(G_\ell, \mathbb{Z}/\ell\mathbb{Z})$$

It follows that  $G_\ell$  is a so called Demushkin group [Dm, L]. These are classified. Since  $\ell$  is odd, the only possibility is

$$G_\ell \cong \langle y_1, \dots, y_{2g} \mid y_1^{\ell^n} [y_1, y_{g+1}] \dots [y_g, y_{2g}] \rangle$$

for some integer  $n \geq 0$ . When  $n > 0$ ,  $G_\ell/[G_\ell, G_\ell]$  has torsion contrary to what was shown above. Therefore  $n = 0$  and the theorem is proved in this case.

Now we turn to the remaining case where  $r \notin [F, F]$ . Let  $\bar{r} \in F/[F, F] \cong \mathbb{Z}^d$  be the image of  $r$ . Fix an isomorphism

$$(F/[F, F])/(\bar{r})/(\text{torsion}) \cong \mathbb{Z}^{d-1}$$

and lift the generators on the right to the free group  $F' = F^{d-1}$ . We thus have a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & \mathbb{Z}^d & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ F' & \longrightarrow & \mathbb{Z}^{d-1} & \longleftarrow & \mathbb{Z}^d/(\bar{r}) \end{array}$$

(The vertical arrow from  $F$  to  $F'$  is labeled  $\phi$ )

given by the solid arrows. We can choose a homomorphism  $\phi : F \rightarrow F'$  which completes the commutative diagram as indicated. Let  $K$  be the quotient of  $F'$  by

the normal subgroup generated by  $r' = \phi(r)$ , and let  $H = \hat{K}$ . The homomorphism  $\phi$  induces a continuous homomorphism  $f : G \rightarrow H$ .

Let  $S_1$  be the set of primes  $\ell$  such that  $\mathbb{Z}^d/(\bar{r})$  has  $\ell$ -torsion. Equivalently  $S_1$  is the minimal set of primes such that  $(\mathbb{Z}^d/(\bar{r}))_\ell$  is torsion free whenever  $\ell \notin S_1$ . We assume that  $\ell \notin S_1$  for the remainder of this paragraph. We claim that  $f$  induces an isomorphism  $G_\ell \cong H_\ell$ . By construction, we have  $H^1(H_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z})$ . If we can show that there is an injection on  $H^2$ , the claim will follow from lemma 4.2. We split this into subcases. Suppose that  $r' = 1$ , then  $H_\ell$  is a free pro- $\ell$  group. Therefore  $H^2(H_\ell) = 0$  so the claim follows in this case. But in fact, this case is impossible because  $G_\ell$  cannot be free. Thus  $r' \neq 1$ . Then

$$H^2(H_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(R', \mathbb{Z}/\ell\mathbb{Z})^{\hat{F}'_\ell} \cong \mathbb{Z}/\ell\mathbb{Z}$$

where  $R' \subset \hat{F}'_\ell$  is the closed normal subgroup generated by  $r'$  (cf [S2, pp 30-31]). We have a similar description for  $H^2(G_\ell)$ . From this it follows easily that the map  $H^2(H_\ell) \rightarrow H^2(G_\ell)$  is nonzero, and therefore an isomorphism.

With the claim now proven, we can work with  $H$  instead of  $G$  provided we choose  $\ell \notin S_1$ . By construction,  $r' \in [F', F']$ , so we are now in the same situation as case 1. The arguments for that case show there is a finite set of primes  $S_2$ , explicitly determined by  $r'$ , such that when  $\ell \notin S_2$ , we have  $H_\ell \cong \hat{\Gamma}_{g,1;\ell}$  for some  $g$ . In conclusion, the theorem holds in the second case when  $S = S_1 \cup S_2$ .  $\square$

**Corollary 4.3.** *With the notation as in theorem 4.1, the maximal pro-nilpotent prime to  $S$  quotients of  $G$  and  $\hat{\Gamma}_{g,1}$  are isomorphic.*

*Proof.* This follows from the theorem and [LO, lemma 2.10].  $\square$

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